SAINT-VENANT'S PROBLEM FOR THIN-WALLED TUBES WITH A CIRCULAR AXIS

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A method for analyzing a thin-walled tube acted upon by a bending moment in the plane of its curved axis has been developed by Von Karman [1]. An essentially new feature in his method was the taking into account of the flattening of the cross-section. Numerous contributions to the subject have been published later by other authors; a critical review of these latter contributions is given in [2]. The publications just referred to deal either with problem of raising the accuracy of the results derived by Von Karman for possible application to a wider interval of occurring parameters, or with some other special cases of loading. A general characteristic of all of these publications is the use of minimum principles applicable to approximate expressions for displacements and stresses. A different approach was chosen by Clark and Reissner [3]; they reduce the problem of the tube acted upon by a bending moment in the plane of the curved axis to that of solving Meissner's equation.

The present paper offers a uniform approach to the problem of deformation for a tube free of surface loading but carrying loads of general form along its boundary line. The problem is treated as one of the theory of thin shells. The boundary conditions of the tube ends are satisfied in accordance with Saint-Venant's principle. The notations used are fundamentally identical with those chosen in [4].

1. Consider the tube part bounded by the sections $\phi = 0$ and $\phi = -\phi_0$ (Fig. 1). We introduce into our discussion dislocational displacements [5], i.e. displacements, non-periodical with respect to ϕ, but in correspondence with the periodical components of defomation. In our present problem

$$\mathbf{U}^g = \{\mathbf{U}^\circ + \Omega^\circ \times \mathbf{r}\} \frac{\varphi + \varphi^\circ}{\varphi^\circ} \qquad (1.1)$$

where $\mathbf{r}$ is the radius vector of an arbitrary point of the middle surface of the tube, while the expression within the curved brackets represents the displacement vector of the middle surface considered as a rigid body, with the origin O of the coordinates used as a pole. The components of the vectors

$$\mathbf{U}^\circ = U_x{}^\circ \mathbf{e}_x + U_y{}^\circ \mathbf{e}_y + U_z{}^\circ \mathbf{e}_z, \qquad \Omega^\circ = \Omega_x{}^\circ \mathbf{e}_x + \Omega_y{}^\circ \mathbf{e}_y + \Omega_z{}^\circ \mathbf{e}_z$$

are constant dislocations. Projecting the vector $\mathbf{U}^g$ on the directions connected with the middle surface of the tube (Fig. 1), we obtain

$$\begin{aligned} u^g &= \{\cos\theta\cos\varphi U_x{}^\circ + \cos\theta\sin\varphi U_y{}^\circ - \sin\theta\, U_z{}^\circ - \\ &\quad - (\alpha + \sin\theta)\sin\varphi R_0\Omega_x{}^\circ + (\alpha + \sin\theta)\cos\varphi R_0\Omega_y{}^\circ\}\, \frac{\varphi + \varphi_0}{\varphi_0} \\ w^g &= \{\sin\theta\cos\varphi U_x{}^\circ + \sin\theta\sin\varphi U\ {}^\circ + \cos\theta U_z{}^\circ + \\ &\quad + \cos\theta\sin\varphi R_0\Omega_x{}^\circ - \cos\theta\cos\varphi R_0\Omega_y{}^\circ\}\, \frac{\varphi + \varphi_0}{\varphi_0} \\ v^g &= \{-\sin\varphi U_x{}^\circ + \cos\varphi U_y{}^\circ - \alpha\cos\theta\cos\varphi R_0\Omega_x{}^\circ - \\ &\quad - \alpha\cos\theta\sin\varphi R_0\Omega_y{}^\circ + (1 + \alpha\sin\theta)\, R_0\Omega_z{}^\circ\}\frac{\varphi + \varphi_0}{\varphi_0} \end{aligned} \tag{1.2}$$

Substituting into (1.2), consecutively, $\phi = 0$ and $\phi = -\phi_0$, we obtain

$$\begin{aligned} u^g(0) &= \cos\theta U_x{}^\circ - \sin\theta U_z{}^\circ + (\alpha + \sin\theta)\, R_0\Omega_y{}^\circ, \qquad & u^g(-\varphi_0) = 0 \\ w^g(0) &= \sin\theta U_x{}^\circ + \cos\theta U_z{}^\circ - \cos\theta R_0\Omega\ {}^\circ, & v^g(-\varphi_0) = 0 \\ v^g(0) &= U_y{}^\circ - \alpha\cos\theta R_0\Omega_x{}^\circ + (1 + \alpha\sin\theta)\, R_0\Omega_z{}^\circ, & w^g(-\varphi_0) = 0 \end{aligned} \tag{1.3}$$

For the horizontal and vertical displacements

$$\Delta_\rho = u\cos\theta + w\sin\theta, \qquad \Delta_z = -u\sin\theta + w\cos\theta \tag{1.4}$$

we consequently find

$$\begin{aligned} \Delta_\rho{}^g(0) &= U_x{}^\circ + \alpha\cos\theta R_0\Omega_y{}^\circ, \qquad & \Delta_\rho{}^g(-\varphi_0) = 0 \\ \Delta_z{}^g(0) &= U_z{}^\circ - (1 + \alpha\sin\theta)\, R_0\Omega_y{}^\circ, & \Delta_z{}^g(-\varphi_0) = 0 \end{aligned} \tag{1.5}$$

Thus, we see that the introduced dislocational displacements determine the displacement of the cross-section $\phi = 0$, considered as a rigid body, with respect to the fixed cross-section $\phi = -\phi_0$ This displacement is characterized by the six parameters $U_x{}^\circ$, $U_y{}^\circ$, $U_z{}^\circ$, $\Omega_x{}^\circ$, $\Omega_y{}^\circ$, $\Omega_z{}^\circ$. If the displacement of the section $\phi = 0$ is to be characterized by the vector $\mathbf{U}^{\circ\circ}\{U_x{}^{\circ\circ}, U_y{}^{\circ\circ}, U_z{}^{\circ\circ}\}$ of displacement of its center and the vector $\Omega^{\circ\circ}\{\Omega_x{}^{\circ\circ}, \Omega_y{}^{\circ\circ}, \Omega_z{}^{\circ\circ}\}$ of rotation angles, then, as easily concluded from Fig. 1

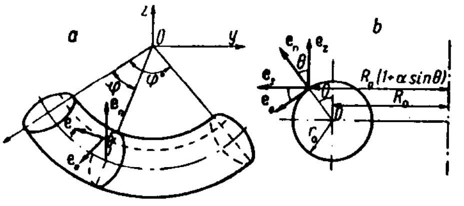

FIG. 1.

$$U_x^{\circ\circ} = U_x^\circ$$
$$U_y^{\circ\circ} = U_y^\circ + R_0\Omega_z^\circ$$
$$U_z^{\circ\circ} = U_z^\circ - R_0\Omega_y^\circ$$
$$\Omega_x^{\circ\circ} = \Omega_x^\circ, \qquad \Omega_y^{\circ\circ} = \Omega_y^\circ$$
$$\Omega_z^{\circ\circ} = \Omega_z^\circ \tag{1.6}$$

By means of well-known formulas we now determine the dislocational components of deformation corresponding to the displacements (1.2)

$$\varepsilon_2^g = \frac{1}{R_0\varphi_0(1+\alpha\sin\theta)}\{-\sin\varphi U_x^\circ + \cos\varphi U\ ^\circ - \alpha\cos\theta\cos\varphi R_0\Omega_x^\circ - \\ - \alpha\cos\theta\sin\varphi R_0\Omega_y^\circ + (1+\alpha\sin\theta) R_0\Omega_z^\circ\} \tag{1.7}$$

$$\omega^g = \frac{1}{R_0\varphi_0(1+\alpha\sin\theta)}\{\cos\theta\cos\varphi U_x^\circ + \cos\theta\sin\varphi U_y^\circ - \sin\theta U_z^\circ - \\ - (\alpha+\sin\theta)\sin\varphi R_0\Omega_x^\circ + (\alpha+\sin\theta)\cos\varphi R_0\Omega_y^\circ\}$$

$$\varkappa_2^g = \frac{1}{R_0^2\varphi_0(1+\alpha\sin\theta)^2}\{\sin\theta\sin\varphi U_x^\circ - \sin\theta\cos\varphi U_y^\circ - \\ - (2+\alpha\sin\theta)\cos\theta\cos\varphi R_0\Omega_x^\circ - \\ - (2+\alpha\sin\theta)\cos\theta\sin\varphi R_0\Omega_y^\circ + (1+\alpha\sin\theta)\sin\theta R_0\Omega_z^\circ\}$$

$$\tau^g = \frac{1}{R_0^2\varphi_0(1+\alpha\sin\theta)^2}\{\sin\theta\cos\theta\cos\varphi U_x^\circ + \sin\theta\cos\theta\sin\varphi U_y^\circ + \cos^2\theta U_z^\circ - \\ - (\alpha+\sin\theta)\sin\theta\sin\varphi R_0\Omega_x^\circ + (\alpha+\sin\theta)\sin\theta\cos\varphi R_0\Omega_y^\circ\}$$

$$\varepsilon_1^g = \varkappa_1^g = 0$$

If, with the aid of the relationships of Hooke's law in its generalized form, we derive from the expressions (1.7) the forces and moments, and from the latter the corresponding surface loading, this loading will be, in general, different from zero. Therefore, we assume expressions of the form

$$u = u^g + u^k, \qquad v = v^g + v^k, \qquad w = w^g + w^k$$

for the total displacements. In doing so we note that the periodical correcting displacements u^k, v^k, w^k will have to be determined from the condition that the total displacements satisfy the problem formulated above.

On the basis of the statico-geometric analogy [6] we construct the solution of the system of equilibrium equations (replacing here the membrane solution otherwise commonly used)

$$T_1^* = \frac{Eh^2}{\nu R_0^2\varphi_0 (1+\alpha\sin\theta)^2}\{\sin\theta\sin\varphi\overline{U}_x^\circ - \sin\theta\cos\varphi\overline{U}_y^\circ - (2+\alpha\sin\theta)\cos\theta\cos\varphi R_0\overline{\Omega}_x^\circ - (2+\alpha\sin\theta)\cos\theta\sin\varphi R_0\overline{\Omega}_y^\circ + (1+\alpha\sin\theta)\sin\theta R_0\overline{\Omega}_z^\circ\} \tag{1.8}$$

$$S^* = \frac{Eh^2}{\nu R_0^2\varphi_0 (1+\alpha\sin\theta)^2}\{-\sin\theta\cos\theta\cos\varphi\overline{U}_x^\circ - \sin\theta\cos\theta\sin\varphi\overline{U}_y^\circ - \cos^2\theta\overline{U}_z^\circ + (\alpha+\sin\theta)\sin\theta\sin\varphi R_0\overline{\Omega}_x^\circ - (\alpha+\sin\theta)\sin\theta\cos\varphi R_0\overline{\Omega}_y^\circ\}$$

$$M_1^* = \frac{Eh^2}{\nu R_0\varphi_0 (1+\alpha\sin\theta)}\{\sin\varphi\overline{U}_x^\circ - \cos\varphi\overline{U}_y^\circ + \alpha\cos\theta\cos\varphi R_0\overline{\Omega}_x^\circ + \alpha\cos\theta\sin\varphi R_0\overline{\Omega}_y^\circ - (1+\alpha\sin\theta) R_0\overline{\Omega}_z^\circ\}$$

$$2H^* = \frac{Eh^2}{\nu R_0\varphi_0 (1+\alpha\sin\theta)}\{\cos\theta\cos\varphi\overline{U}_x^\circ + \cos\theta\sin\varphi\overline{U}_y^\circ - \sin\theta\overline{U}_z^\circ - (\alpha+\sin\theta)\cos\varphi R_0\overline{\Omega}_y^\circ - (\alpha+\sin\theta)\sin\varphi R_0\overline{\Omega}_x^\circ\}$$

$$T_2^* = M_2^* = 0, \quad \nu = \sqrt{12(1-\mu^2)}$$

In these equations h is the thickness of the shell. Appropriate choice of the parameters $\overline{U}_x^\circ$, $\overline{U}_y^\circ$, $\overline{U}_z^\circ$, $\overline{\Omega}_x^\circ$, $\overline{\Omega}_y^\circ$, $\overline{\Omega}_z^\circ$ assures periodicity of the correcting displacements u^k, v^k, w^k. Finally, we define the functions X, Y, Z as periodical solutions of the following equations:

$$\frac{d^2X}{d\theta^2} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{dX}{d\theta} + \left[-\frac{\alpha^2\cos^2\theta}{(1+\alpha\sin\theta)^2} + i2d^2\frac{\sin\theta}{1+\alpha\sin\theta}\right]X = -d^2\frac{\cos\theta}{1+\alpha\sin\theta} \tag{1.9}$$

$$\frac{d^2Y}{d\theta^2} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{dY}{d\theta} + \left[-\frac{4\alpha^2\cos^2\theta}{(1+\alpha\sin\theta)^2} + i2d^2\frac{\sin\theta}{1+\alpha\sin\theta}\right]Y = -d^2\frac{\cos\theta}{(1+\alpha\sin\theta)}$$

$$\frac{d^2Z}{d\theta^2} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{dZ}{d\theta} + \left[-\frac{4\alpha^2\cos^2\theta}{(1+\alpha\sin\theta)^2} + i2d^2\frac{\sin\theta}{1+\alpha\sin\theta}\right]Z = -d^2\frac{\sin\theta+\alpha}{(1+\alpha\sin\theta)^2}$$

$$(2d^2 = R_0\alpha^2\nu / h)$$

It is easily seen that the functions X and Y are even and Z is odd with respect to the substitution of $\pi - \theta$ for θ. Integration of the third equation between the limits 0 and 2π leads to the equation

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin\theta Z d\theta}{1+\alpha\sin\theta} = 0 \tag{1.10}$$

valid within the limits of the theory of thin shells. We introduce the notations

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi}\cos\theta X d\theta, \qquad I_2 = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\theta Y d\theta}{1+\alpha\sin\theta}, \qquad I_3 = \frac{1}{2\pi}\int_0^{2\pi}\frac{Z d\theta}{1+\alpha\sin\theta} \tag{1.11}$$

$$J_k(\alpha) = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(1+\alpha\sin\theta)^k}$$

$$\Lambda_1 = 2\frac{(\mathrm{Re}\, I_1)^2 + (\mathrm{Im}\, I_1)^2}{\mathrm{Re}\, I_1}, \quad \Lambda_2 = 2\frac{(\mathrm{Re}\, I_2)^2 + (\mathrm{Im}\, I_2)^2}{\mathrm{Re}\, I_2}, \quad \Lambda_3 = 2\alpha\frac{(\mathrm{Re}\, I_3)^2 + (\mathrm{Im}\, I_3)^2}{\mathrm{Re}\, I_3} \tag{1.12}$$

The relations (1.7) show that the components of the dislocational deformation represent very simple functions of the angle ϕ. This predetermines the form of the correcting displacements. In accordance with the procedure to be used we subdivide the problem into three special problems.

2. The case corresponding to U_z° shall be called the first symmetrical problem. From (1.7) we conclude that the non-vanishing deformation components ω^g, τ^g are independent of ϕ. This case corresponds to the symmetrical torsion of a shell of revolution. The solution is elementary and we can confine ourselves to stating the final results.

For the torque and the shearing force we find

$$M_\varphi = R_0 P_z = \frac{EJ}{R_0^2}\left\{\frac{1}{(1+\mu)\varphi_0}\frac{[J_2(\alpha)]^2}{J_3(\alpha)}\right\} U_z^\circ \tag{2.1}$$

where $J = \pi r_0^3 h$ = equatorial moment of inertia of the cross-section of the tube.

The shear stress is

$$\sigma_{12} = \sigma_0\left\{\frac{1}{J_2(\alpha)}\frac{1}{(1+\alpha\sin\theta)^2}\right\} \qquad \left(\sigma_0 = \frac{M_\varphi}{2J/r_0}\right) \tag{2.2}$$

where σ_0 is the maximum stress according to the elementary theory of torsion.

$$v^k = \alpha(1+\alpha\sin\theta)\left\{\left[\int_{\theta_0}^{\theta}\frac{\sin\theta d\theta}{(1+\alpha\sin\theta)^2} + \frac{\alpha J_2(\alpha)}{J_3(\alpha)}\int_{\theta_0}^{\theta}\frac{d\theta}{(1+\alpha\sin\theta)^3}\right]\frac{U_z^\circ}{\varphi_0} + R_0 C_1\right\} \tag{2.3}$$

3. The case corresponding to Ω_z° shall be called the second symmetrical case. Inasmuch as, in accordance with (1.7), ϵ_2^g and κ_2^g are independent of ϕ, we use for the solution of the problem Meissner's equation in transformed form

$$\frac{d^2\vartheta^{\vee k}}{d\theta^2}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{d\vartheta^{\vee k}}{d\theta}+\left[-\frac{\alpha^2\cos^2\theta}{(1+\alpha\sin\theta)^2}+i2d^2\frac{\sin\theta}{1+\alpha\sin\theta}\right]\vartheta^{\vee k}=$$
$$=i2d^2\frac{\sin\theta}{1+\alpha\sin\theta}\left\{\vartheta^g+i\frac{\nu}{Eh^2}\vartheta^*\right\}$$

where

$$\vartheta^g=\frac{1}{\alpha\sin\theta}\left[\frac{d(1+\alpha\sin\theta)\varepsilon_2{}^g}{d\theta}-\alpha\cos\theta\,\varepsilon_1{}^g\right]=\frac{\Omega_z{}^\circ}{\varphi_0}\cot\theta \tag{3.1}$$
$$\vartheta^*=-\frac{1}{\alpha\sin\theta}\left[\frac{d(1+\alpha\sin\theta)M_1{}^*}{d\theta}-\alpha\cos\theta\,M_2{}^*\right]\frac{Eh^2\overline{\Omega}_z{}^\circ}{\nu\varphi_0}\cot\theta$$

Comparing the obtained equation with the first equation of the system (1.9), we find

$$\vartheta^{\vee k}=\frac{2d^2}{\varphi_0}(\overline{\Omega}_z{}^\circ-i\Omega_z{}^\circ)X \tag{3.2}$$

The forces and moments express themselves in terms of the function X by means of the following relations:

$$T_1=\frac{Eh^2}{\nu r_0}\left\{\frac{2\overline{\Omega}_z{}^\circ}{\varphi_0}\left(\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\operatorname{Im}X+\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\right)-\frac{2\Omega_z{}^\circ}{\varphi_0}\frac{\alpha\cos\theta\operatorname{Re}X}{1+\alpha\sin\theta}\right\}$$
$$T_2=\frac{Eh^2}{\nu r_0}\left\{\frac{2\overline{\Omega}_z{}^\circ}{\varphi_0}\operatorname{Im}\frac{dX}{d\theta}-2\frac{\Omega_z{}^\circ}{\varphi_0}\operatorname{Re}\frac{dX}{d\theta}\right\} \tag{3.3}$$
$$M_1=\frac{Eh^3}{\nu^2r_0}\left\{\frac{2\overline{\Omega}_z{}^\circ}{\varphi_0}\operatorname{Re}\left(\frac{dX}{d\theta}+\mu\frac{\alpha\cos\theta X}{1+\alpha\sin\theta}\right)+\right.$$
$$\left.+\frac{2\Omega_z{}^\circ}{\varphi_0}\left[\operatorname{Im}\left(\frac{dX}{d\theta}+\mu\frac{\alpha\cos\theta X}{1+\alpha\sin\theta}\right)+\frac{\mu}{2}\frac{\alpha\sin\theta}{1+\alpha\sin\theta}\right]\right\}$$
$$M_2=\frac{Eh^3}{\nu^2r_0}\left\{\frac{2\overline{\Omega}_z{}^\circ}{\varphi_0}\operatorname{Re}\left(\mu\frac{dX}{d\theta}+\frac{\alpha\cos\theta X}{1+\alpha\sin\theta}\right)+\right.$$
$$\left.+\frac{2\Omega_z{}^\circ}{\varphi_0}\left[\operatorname{Im}\left(\mu\frac{dX}{d\theta}+\frac{\alpha\cos\theta X}{1+\alpha\sin\theta}\right)+\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\right]\right\}$$

The vertical displacement is

$$\Delta_z{}^k=r_0\left(C_2-\int_{\theta^\circ}^{\theta}\cos\theta\operatorname{Re}\vartheta^{\vee k}d\theta\right) \tag{3.4}$$

It is easily seen that the condition of periodicity of $\Delta_z{}^k$ assumes, in accordance with (3.2) and (1.11), the form

$$\overline{\Omega}_z{}^\circ=-\frac{\operatorname{Im}I_1}{\operatorname{Re}I_1}\Omega_z{}^\circ \tag{3.5}$$

Eliminating Ω_z° from the relations (3.3), we easily find with the aid of the aforementioned properties of the function X that the forces and moments, applied at an arbitrary cross-section (essentially T_2), are statically equivalent to the bending moment

$$M_z = \frac{EJ}{R_0^2}\left\{\frac{\Lambda_1}{d^2\varphi_0}\right\} R_0\Omega_z^\circ \tag{3.6}$$

Introducing the membrane and bending stresses

$$\sigma_1^m = \frac{T_1}{h}, \qquad \sigma_2^m = \frac{T_2}{h}, \qquad \sigma_1^b = \frac{6M_1}{h^2}, \qquad \sigma_2^b = \frac{6M_2}{h^2} \tag{3.7}$$

we find

$$\frac{\sigma_1^m}{\sigma_0} = \frac{1}{\Lambda_1}\left\{-\frac{\alpha\cos\theta \operatorname{Re} X}{1+\alpha\sin\theta} - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\left(\frac{\alpha\cos\theta\operatorname{Im} X}{1+\alpha\sin\theta} + \frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\right)\right\} \tag{3.8}$$

$$\frac{\sigma_2^m}{\sigma_0} = \frac{1}{\Lambda_1}\left\{-\operatorname{Re}\frac{dX}{d\theta} - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Im}\frac{dX}{d\theta}\right\} \qquad \left(\sigma_0 = \frac{M_z}{J/r_0}\right)$$

$$\frac{\sigma_1^b}{\sigma_0} = \frac{6}{\nu\Lambda_1}\left\{\operatorname{Im}\left(\frac{dX}{d\theta} + \mu\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right) + \frac{\mu\alpha\sin\theta}{2(1+\alpha\sin\theta)} - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Re}\left(\frac{dX}{d\theta} + \frac{\mu\alpha\cos\theta}{1+\alpha\sin\theta}X\right)\right\}$$

$$\frac{\sigma_2^b}{\sigma_0} = \frac{6}{\nu\Lambda_1}\left\{\operatorname{Im}\left(\mu\frac{dX}{d\theta} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right) + \frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)} - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Re}\left(\mu\frac{dX}{d\theta} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right)\right.$$

where $\sigma_0 = M_z/(J/r_0)$ represents the maximum stress according to the elementary theory of bending, and finally

$$\begin{aligned}\Delta_z^k &= r_0\left\{C_2 + \left[-\operatorname{Im}\int_{\theta^\circ}^{\theta}\cos\theta X d\theta + \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Re}\int_{\theta_0}^{\theta}\cos\theta X\,d\theta\right]\frac{2\Omega_z^\circ}{\varphi_0}\right\}\\ \Delta_\rho^k &= (1+\alpha\sin\theta)\left\{-1 - \frac{2h}{\nu R_0}\left(\operatorname{Re}\frac{dX}{d\theta} + \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Im}\frac{dX}{d\theta}\right)\right\}\frac{2R_0\Omega_z^\circ}{\varphi_0}\end{aligned} \tag{3.9}$$

4. The cases characterized by U_y°, $\Omega_x^\circ \neq 0$ and U_x°, $\Omega_y^\circ \neq 0$. shall be called the first and the second inverse symmetrical problems.

With some generalization of the results derived in [7], we find that the basic difficulty encountered in the case under consideration consists in the solution of the equation

$$\frac{d^2\chi^{\vee k}}{d\theta^2} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{d\chi^{\vee k}}{d\theta} + \left[-\frac{4\alpha^2\cos^2\theta}{(1+\alpha\sin\theta)^2} + i2d^2\frac{\sin\theta}{1+\alpha\sin\theta}\right]\chi^{\vee k} = i2d^2\frac{\sin\theta}{1+\alpha\sin\theta}\left(\chi^g + i\frac{\nu}{Eh^2}\chi^*\right)$$

where

$$\chi^g = \frac{1+\alpha\sin\theta}{\alpha}\frac{d\varepsilon_{2,1}^{g}}{d\theta} - \cot\theta\varepsilon^g_{1,1} \mp \frac{1}{\sin\theta}\omega^g_{,1} \tag{4.1}$$

$$\chi^{*} = -\frac{1+\alpha\sin\theta}{\alpha}\frac{dM_{1,1}^{*}}{d\theta} + \cot\theta M_{2,1}^{*} \pm \frac{1}{\sin\theta}2H_{,1}^{*}$$

The subscript 1 after a comma signifies here and in the following that the quantity concerned represents a multiplier of cos ϕ or sin ϕ in the corresponding expression. Furthermore, if there are two functions (signs), the upper one refers to the first inverse-symmetrical case, and the lower to the second one.

Substituting into (4.1) the corresponding expressions, we find, with the aid of (1.9),

$$\chi^{\vee k} = \left[-\frac{4}{R_0\varphi_0}\,{}^{\overline{U}_y{}^{\circ}}_{-\overline{U}_x{}^{\circ}} + i\frac{4}{R_0\varphi_0}\,{}^{U_y{}^{\circ}}_{-U_x{}^{\circ}}\right]Y + \left[\frac{4}{\varphi_0}\,{}^{\overline{\Omega}_x{}^{\circ}}_{\overline{\Omega}_y{}^{\circ}} - i\frac{4}{\varphi_0}\,{}^{\Omega_x{}^{\circ}}_{\Omega_y{}^{\circ}}\right]Z \tag{4.2}$$

The forces and moments express themselves in terms of the fundamental complex function by the relations

$$T_{1,1} = \frac{Eh^2}{\nu r_0}\left\{-\frac{4}{R_0\varphi_0}\left[\frac{\alpha\cos\theta\,\mathrm{Im}\,Y}{1+\alpha\sin\theta} + \frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)^2}\right]{}^{\overline{U}_y{}^{\circ}}_{-\overline{U}_x{}^{\circ}} + \right. \tag{4.3}$$

$$\left. + \frac{4}{R_0\varphi_0}\frac{\alpha\cos\theta\,\mathrm{Re}\,Y}{1+\alpha\sin\theta}\,{}^{U_y{}^{\circ}}_{-U_x{}^{\circ}} + \frac{4}{\varphi_0}\left[\frac{\alpha\cos\theta\,\mathrm{Im}\,Z}{1+\alpha\sin\theta} - \frac{\alpha\cos\theta}{2(1+\alpha\sin\theta)^2}\right]{}^{\overline{\Omega}_x{}^{\circ}}_{\overline{\Omega}_y{}^{\circ}} - \frac{4}{\varphi_0}\frac{\alpha\cos\theta\,\mathrm{Re}\,Z}{1+\alpha\sin\theta}\,{}^{\Omega_x{}^{\circ}}_{\Omega_g{}^{\circ}}\right\}$$

$$T_{2,1} = \frac{Eh^2}{\nu r_0}\left\{-\frac{4}{R_0\varphi_0}\mathrm{Im}\left(\frac{dY}{d\theta} + \frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right){}^{\overline{U}_y{}^{\circ}}_{-\overline{U}_x{}^{\circ}} + \right.$$

$$+ \frac{4}{R_0\varphi_0}\mathrm{Re}\left(\frac{dY}{d\theta} + \frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right){}^{\overline{U}_y{}^{\circ}}_{-\overline{U}_x{}^{\circ}} + \frac{4}{\varphi_0}\mathrm{Im}\left(\frac{dZ}{d\theta} + \frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right){}^{\Omega_x{}^{\circ}}_{\overline{\Omega}_y{}^{\circ}} -$$

$$\left. - \frac{4}{\varphi_0}\mathrm{Re}\left(\frac{dZ}{d\theta} + \frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right){}^{\Omega_x{}^{\circ}}_{\Omega_u{}^{\circ}}\right\}$$

$$S_{,1} = \frac{Eh^2}{\nu r_0}\left\{-\frac{4}{R_0\varphi_0}\frac{\alpha\,\mathrm{Im}\,Y}{1+\alpha\sin\theta}\,{}^{\overline{U}_y{}^{\circ}}_{\overline{U}_x{}^{\circ}} + \frac{4}{R_0\varphi_0}\frac{\alpha\,\mathrm{Re}\,Y}{1+\alpha\sin\theta}\,{}^{U_y{}^{\bullet}}_{U_x{}^{\circ}} + \frac{4}{\varphi_0}\frac{\alpha\,\mathrm{Im}\,Z}{1+\alpha\sin\theta}\,{}^{\overline{\Omega}_x{}^{\circ}}_{\overline{\Omega}_y{}^{\circ}} - \right.$$

$$\left. - \frac{4}{\varphi_0}\frac{\alpha\,\mathrm{Re}\,Z}{(1+\alpha\sin\theta)}\,{}^{\Omega_x{}^{\circ}}_{-\Omega_y{}^{\circ}}\right\}$$

$$M_{1,1}=\frac{Eh^3}{\nu^2 r_0}\frac{4}{R_0\varphi_0}\Big\{-\mathrm{Re}\Big(\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\Big)\begin{matrix}\bar U_y^\circ\\ -\bar U_x^\circ\end{matrix}-$$

$$-\Big[\mathrm{Im}\Big(\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\Big)+\frac{\mu\alpha\sin\theta}{2(1+\alpha\sin\theta)}\Big]\begin{matrix}U_y^\circ\\ -U_x^\circ\end{matrix}+$$

$$+\mathrm{Re}\Big(\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\Big)\begin{matrix}R_0\bar\Omega_x^\circ\\ R_0\bar\Omega_y^\circ\end{matrix}+\Big[\mathrm{Im}\Big(\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\Big)-$$

$$-\frac{\mu\alpha\cos\theta}{2(1+\alpha\sin\theta)}\Big]\begin{matrix}R_0\Omega_x^\circ\\ R_0\Omega_y^\circ\end{matrix}\Big\}$$

$$M_{2,1}=\frac{Eh^3}{\nu^2 r_0}\frac{4}{R_0\varphi_0}\Big\{-\mathrm{Re}\Big(\mu\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\Big)\begin{matrix}\bar U_y^\circ\\ -\bar U_x^\circ\end{matrix}-$$

$$-\Big[\mathrm{Im}\Big(\mu\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\Big)+\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\Big]\begin{matrix}U_y^\circ\\ -U_x^\circ\end{matrix}+$$

$$+\mathrm{Re}\Big(\mu\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{(1+\alpha\sin\theta)}\Big)\begin{matrix}R_0\bar\Omega_x^\circ\\ R_0\bar\Omega_y^\circ\end{matrix}+$$

$$+\Big[\mathrm{Im}\Big(\mu\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\Big)-\frac{\alpha\cos\theta}{2(1+\alpha\sin\theta)}\Big]\begin{matrix}R_0\Omega_x^\circ\\ R_0\Omega_y^\circ\end{matrix}\Big\}$$

$$H_{,1}=\frac{Eh^3}{\nu^2 r_0}(1-\mu)\frac{4}{R_0\varphi_0}\frac{\alpha}{1+\alpha\sin\theta}\Big\{\mathrm{Re}\,Y\begin{matrix}\bar U_y^\circ\\ \bar U_x^\circ\end{matrix}+\mathrm{Im}\,Y\begin{matrix}U_y^\circ\\ U_x^\circ\end{matrix}-$$

$$-\mathrm{Re}\,Z\begin{matrix}\mathrm{Re}\,\bar\Omega_x^\circ\\ -R_0\bar\Omega_y^\circ\end{matrix}-\mathrm{Im}\,Z\begin{matrix}R_0\Omega_x^\circ\\ -R_0\Omega_y^\circ\end{matrix}\Big\}$$

Finally, the displacements can be written in the form

$$\Delta_{z,1}^{\ k}=r_0(1+\alpha\sin\theta)\Big\{-\int_{\theta_0}^{\theta}\frac{\cos\theta\,\mathrm{Re}\,\chi^{\vee k}}{1+\alpha\sin\theta}d\theta+1\begin{matrix}C_3\\ C_4\end{matrix}\Big\} \tag{4.4}$$

$$\Delta_{\rho,1}^{\ k}=r_0\Big\{\alpha\cos\theta\int_{\theta_0}^{\theta}\frac{\cos\theta\,\mathrm{Re}\,\chi^{\vee k}}{1+\alpha\sin\theta}d\theta-\int_{\theta_0}^{\theta}\frac{(\sin\theta+\alpha)\,\mathrm{Re}\,\chi^{\vee k}}{1+\alpha\sin\theta}d\theta-\alpha\cos\theta\begin{matrix}C_3\\ C_4\end{matrix}+1\begin{matrix}C_5\\ C_6\end{matrix}\Big\} \tag{4.5}$$

The conditions of periodicity of $\Delta_{z\ 1}$ and $\Delta_{\rho,1}$ are, according to (4.2) and (1.11)

$$\begin{matrix}\bar U_y^\circ\\ \bar U_x^\circ\end{matrix}=-\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\begin{matrix}U_y^\circ\\ U_x^\circ\end{matrix},\qquad \begin{matrix}\bar\Omega_x^\circ\\ \bar\Omega_y^\circ\end{matrix}=-\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\begin{matrix}\Omega_x^\circ\\ \Omega_y^\circ\end{matrix} \tag{4.6}$$

and by virtue of these expressions we have

$$\Delta_{z,1}^{k} = (1+\alpha\sin\theta)\left\{\left(\int_{\theta_0}^{\theta}\frac{\cos\theta\,\mathrm{Im}\,Y}{1+\alpha\sin\theta}d\theta - \frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\int_{\theta_0}^{\theta}\frac{\cos\theta\,\mathrm{Re}\,Y}{1+\alpha\sin\theta}d\theta\right)\frac{4\alpha}{\varphi_0}\begin{matrix}U_y^{\circ}\\ -U_x^{\circ}\end{matrix} - \right.$$

$$\left. - \left(\int_{\theta_0}^{\theta}\frac{\cos\theta\,\mathrm{Im}\,Z}{1+\alpha\sin\theta}d\theta - \frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\int_{\theta_0}^{\theta}\frac{\cos\theta\,\mathrm{Re}\,Z}{1+\alpha\sin\theta}d\theta\right)\frac{4\alpha}{\varphi_0}\begin{matrix}R_0\Omega_x^{\circ}\\ R_0\Omega_y^{\circ}\end{matrix} + r_0\begin{matrix}C_3\\ C_4\end{matrix}\right\} \qquad (4.7)$$

$$\Delta_{\rho,1}^{k} = \left[\left(\int_{\theta_0}^{\theta}\frac{(\sin\theta+\alpha)\,\mathrm{Im}\,Y}{1+\alpha\sin\theta}d\theta - \cos\theta\int_{\theta_0}^{\theta}\frac{\alpha\cos\theta\,\mathrm{Im}\,Y}{1+\alpha\sin\theta}d\theta\right) - \right.$$

$$\left. - \frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\left(\int_{\theta_0}^{\theta}\frac{(\sin\theta+\alpha)\,\mathrm{Re}\,Y}{1+\alpha\sin\theta}\alpha\theta - \cos\theta\int_{\theta_0}^{\theta}\frac{\alpha\cos\theta\,\mathrm{Re}\,Y}{1+\alpha\sin\theta}d\theta\right)\right]\frac{4\alpha}{\varphi_0}\begin{matrix}U_y^{\circ}\\ -U_x^{\circ}\end{matrix} -$$

$$-\left[\left(\int_{\theta_0}^{\theta}\frac{(\sin\theta+\alpha)\,\mathrm{Im}\,Z}{1+\alpha\sin\theta}d\theta - \cos\theta\int_{\theta_0}^{\theta}\frac{\alpha\cos\theta\,\mathrm{Im}\,Z}{1+\alpha\sin\theta}d\theta\right) - \frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\,\cdot\right.$$

$$\left.\left(\int_{\theta_0}^{\theta}\frac{(\sin\theta+\alpha)\,\mathrm{Re}\,Z}{1+\alpha\sin\theta}d\theta - \cos\theta\int_{\theta_0}^{\theta}\frac{\alpha\cos\theta\,\mathrm{Re}\,Z}{1+\alpha\sin\theta}d\theta\right)\right]\frac{4\alpha}{\varphi_0}\begin{matrix}R_0\Omega_x^{\circ}\\ R_0\Omega_y^{\circ}\end{matrix} - r_0\cos\theta\begin{matrix}C_3\\ C_4\end{matrix} + r_0\begin{matrix}C_5\\ C_6\end{matrix} \pm$$

$$\pm v,_1^k = \frac{1}{\varphi_0}\left\{-1\begin{matrix}U_y^{\circ}\\ -U_x^{\circ}\end{matrix} + \alpha\cos\theta\begin{matrix}R_0\Omega_x^{\circ}\\ R_0\Omega_y^{\circ}\end{matrix}\right\} - \Delta_{\rho,1}$$

Using, furthermore, the aforementioned properties of the functions Y and Z, the relation (1.10) and the notations (1.13), we find, with the aid of (4.3)

$$R_0P_\rho = \frac{EJ}{R_0^2}\left\{\frac{2\Lambda_2}{d^2\varphi_0}\right\}\begin{matrix}U_y^{\circ}\sin\varphi_0,\\ U_x^{\circ}\cos\varphi_0,\end{matrix} \qquad M_\varphi = \frac{EJ}{R_0^2}\left\{\frac{2\Lambda_3}{d^2\varphi_0}\right\}\begin{matrix}-R_0\Omega_x^{\circ}\sin\varphi\\ R_0\Omega_y^{\circ}\cos\varphi\end{matrix}$$

$$R_0P_\varphi = \frac{EJ}{R_0^2}\left\{\frac{2\Lambda_2}{d^2\varphi_0}\right\}\begin{matrix}U_y^{\circ}\cos\varphi\ ,\\ -U_x^{\circ}\sin\varphi,\end{matrix} \qquad M_\rho = \frac{EJ}{R_0^2}\left\{\frac{2\Lambda_3}{d^2\varphi_0}\right\}\begin{matrix}R_0\Omega_x^{\circ}\cos\varphi\\ R_0\Omega_y^{\circ}\sin\varphi\end{matrix} \qquad (4.8)$$

$$M_z = \frac{EJ}{R_0^2}\left\{\frac{2\Lambda_2}{d^2\varphi_0}\right\}\begin{matrix}-U_y^{\circ}\cos\varphi,\\ U_x^{\circ}\sin\varphi\ ,\end{matrix} \qquad P_z = 0 \qquad (4.9)$$

The maximum stresses in the extreme filaments of the tube are, according to the elementary theory of bending

$$\sigma_{0z} = \frac{M_{z,1}}{J/r_0} = -\frac{4Eh\Lambda_2}{\nu r_0R_0\varphi_0}\begin{matrix}U_y^{\circ}\\ -U_x^{\circ}\end{matrix}, \qquad \sigma_{0\rho} = \frac{M_{\rho 1}}{J/r_0} = \frac{4Eh\Lambda_3}{\nu r_0\varphi_0}\begin{matrix}\Omega_x^{\circ}\\ \Omega_y^{\circ}\end{matrix} \qquad (4.10)$$

Introducing the stresses

$$\sigma_1^{(1)} = \frac{T_1}{h}, \qquad \sigma_2^{(1)} = \frac{T_2}{h}, \qquad \sigma_{12}^{(1)} = \frac{S}{h}$$

$$\sigma_1^{(2)} = \frac{6M_1}{h^2}, \qquad \sigma_2^{(2)} = \frac{6M_2}{h^2}, \qquad \sigma_{12}^{(2)} = \frac{6H}{h^2}$$

we find for them, on the basis of (4.3), the following expressions:

(4.11)

$$\sigma_1^{(1)} = \sigma_{0z}\left\{-\frac{1}{\Lambda_2}\left[\frac{\alpha\cos\theta\operatorname{Re}Y}{1+\alpha\sin\theta}+\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\left(\frac{\alpha\cos\theta\operatorname{Im}Y}{1+\alpha\sin\theta}+\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)^2}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}+$$

$$+\sigma_{0\rho}\left\{-\frac{1}{\Lambda_3}\left[\frac{\alpha\cos\theta\operatorname{Re}Z}{1+\alpha\sin\theta}+\frac{\operatorname{Im}I_3}{\operatorname{Re}I_3}\left(\frac{\alpha\cos\theta\operatorname{Im}Z}{1+\alpha\sin\theta}-\frac{\alpha\cos\theta}{2(1+\alpha\sin\theta)^2}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}+$$

$$+\sigma_2^{(1)} = \sigma_{0z}\left\{-\frac{1}{\Lambda_2}\left[\operatorname{Re}\left(\frac{dY}{d\theta}+\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right)+\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\operatorname{Im}\left(\frac{dY}{d\theta}+\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \cos\varphi\end{matrix}\right\}+$$

$$+\sigma_{0\rho}\left\{-\frac{1}{\Lambda_3}\left[\operatorname{Re}\left(\frac{dZ}{d\theta}+\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right)+\frac{\operatorname{Im}I_3}{\operatorname{Re}I_3}\operatorname{Im}\left(\frac{dZ}{d\theta}+\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}$$

$$\sigma_{12}^{(1)} = \sigma_{0z}\left\{\mp\frac{1}{\Lambda_2}\left(\operatorname{Re}Y+\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\operatorname{Im}Y\right)\frac{\alpha}{1+\alpha\sin\theta}\begin{matrix}\sin\varphi\\ \cos\varphi\end{matrix}\right\}+$$

$$+\sigma_{0\rho}\left\{\mp\frac{1}{\Lambda_3}\left(\operatorname{Re}Z+\frac{\operatorname{Im}I_3}{\operatorname{Re}I_3}\operatorname{Im}Z\right)\frac{\alpha}{1+\alpha\sin\theta}\begin{matrix}\sin\varphi\\ \cos\varphi\end{matrix}\right\}$$

$$\sigma_1^{(2)} = \sigma_{0z}\left\{\frac{\sigma}{\nu\Lambda_2}\left[\operatorname{Im}\left(\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right)+\frac{\mu\alpha\sin\theta}{2(1+\alpha\sin\theta)^2}-\right.\right.$$

$$\left.\left.-\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\operatorname{Re}\left(\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}+$$

$$+\sigma_{0\rho}\left\{\frac{6}{\nu\Lambda_3}\left[\operatorname{Im}\left(\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right)-\frac{\mu\alpha\cos\theta}{2(1+\alpha\sin\theta)^2}-\right.\right.$$

$$\left.\left.-\frac{\operatorname{Im}I_3}{\operatorname{Re}I_3}\operatorname{Re}\left(\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}$$

$$\sigma_2^{(2)} = \sigma_{0z}\left\{\frac{6}{\nu\Lambda_2}\left[\operatorname{Im}\left(\mu\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right)+\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)^2}-\right.\right.$$

$$\left.\left.-\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\operatorname{Re}\left(\mu\frac{dY}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Y}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}+$$

$$+\sigma_{0\rho}\left\{\frac{6}{\nu\Lambda_3}\left[\operatorname{Im}\left(\mu\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right.\right.\right.-$$

$$\left.\left.\left.-\frac{\alpha\cos\theta}{2(1+\alpha\sin\theta)^2}-\frac{\operatorname{Im}I_3}{\operatorname{Re}I_3}\operatorname{Re}\left(\mu\frac{dZ}{d\theta}+(1+\mu)\frac{\alpha\cos\theta Z}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}$$

$$\sigma_{12}^{(2)} = \sigma_{0z}\left\{\mp\frac{6(1-\mu)}{\nu\Lambda_2}\left(\operatorname{Im}Y-\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\operatorname{Re}Y\right)\frac{\alpha}{1+\alpha\sin\theta}\begin{matrix}\sin\varphi\\ \cos\varphi\end{matrix}\right\}+$$

$$+\sigma_{0\rho}\left\{\mp\frac{6(1-\mu)}{\nu\Lambda_3}\left(\operatorname{Im}Z-\frac{\operatorname{Im}I_3}{\operatorname{Re}I_3}\operatorname{Re}Z\right)\frac{\alpha}{1+\alpha\sin\theta}\begin{matrix}\sin\varphi\\ \cos\varphi\end{matrix}\right\}$$

5. A series of examples of tubes in bending by a bending moment in the plane of the curved axis of the tube has been studied with the aid of the relations given above. The results obtained were in good agreement with those derived by other authors and with the results, thus far available, of experimental tests. As an example of non-symmetrical bending the case has been investigated of a half ring $\phi_0 = \pi$ under torsion (by means of a moment).

Continued fractions in connection with the method of expansion in terms of the "small" parameter a were used for the solution of Equation (1.9). Figure 2 shows by solid line the axial (small squares) and transverse (small circles) stresses calculated from Formulas (4.11); these stresses are located at the outer surface of the tube characterized by the parameters $d^2 = 21.4$ and $a = 1/3$; the broken line reproduces the results obtained in [8]. Solving (2.1), (3.6), (4.8) and (4.9) for the parameters of the dislocation, we find

$$U_x{}^\circ = \frac{R_0{}^2}{EJ}\{\beta R_0 P_x{}^\circ\},\quad U_y{}^\circ = \frac{R_0{}^2}{EJ}\{\beta R_0 P_y{}^\circ\},\quad R_0\Omega_z{}^\circ = \frac{R_0{}^2}{EJ}\{\gamma R_0 P_y{}^\circ + \gamma M_z{}^\circ\},$$

$$U_z{}^\circ = \frac{R_0{}^2}{EJ}\{\delta R_0 P_z{}^\circ\},\quad R_0\Omega_x{}^\circ = \frac{R_0{}^2}{EJ}\{\chi M_x{}^\circ\},\quad R_0\Omega_y{}^\circ = \frac{R_0{}^2}{EJ}\{\chi M_y{}^\circ - \chi R_0 P_z{}^\circ\} \qquad (5.1)$$

where

$$\beta = \frac{d^2\varphi_0}{2\Lambda_2},\qquad \gamma = \frac{d^2\varphi_0}{\Lambda_1},\qquad \chi = \frac{d^2\varphi_0}{2\Lambda_3},\qquad \delta = (1+\mu)\varphi_0\frac{J_3(\alpha)}{[J_2(\alpha)]^2}$$

The last relations permit the expression, by means of the formulas derived above, of the displacements of the edge of the tube with the

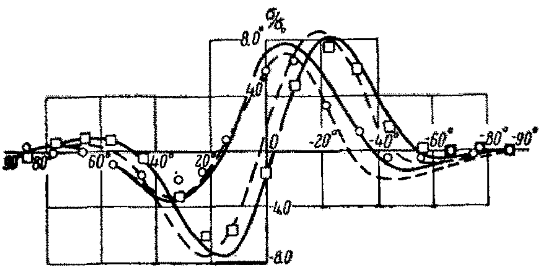

FIG. 2.

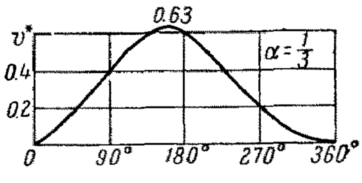

FIG. 3.

aid of the quantities $P_x{}^\circ$, $P_y{}^\circ$, $P_z{}^\circ$, $M_x{}^\circ$, $M_y{}^\circ$, $M_z{}^\circ$ which chacracterize the loading applied at the edge. It is not difficult to see that the correcting and the dislocational displacements are quantities of the same order of magnitude.

Figure 3 shows, as an example, the warping displacement $v^* = U_z{}^0/\phi_0$, corresponding to the first symmetrical case for $a = 1/3$. Figure 4 gives a comparison of the quantity

$$\frac{\gamma}{\varphi_0} = \frac{d^2}{\Lambda_1}$$

which offers a measure of rigidity against bending for the second symmetrical case (case of Von Karman), with values obtained by various authors by means of experimental tests (see [2]). Notwithstanding the fact that the quantity Λ_1 was determined on the basis of the simplifying assumption $a = 0$, the agreement of the theoretical work with the experimental tests is entirely satisfactory.

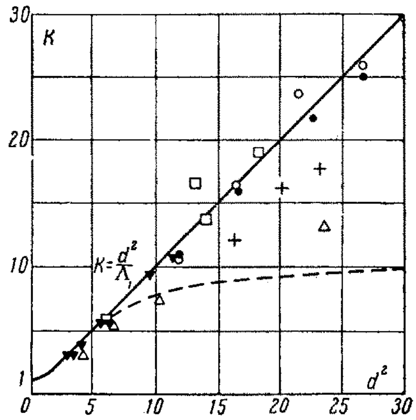

FIG. 4.

As to the other coefficients of rigidity, their magnitude depends essentially on the kind of boundary conditions at the ends of the tube. To establish a solution which satisfies the boundary conditions it will be necessary to use, in addition, a new procedure involving, in particular, solutions which take care of the edge effects. In such cases the discussion presented above can be used as a basis for the derivation of the exact solution.

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